Indian Statistical Institute, Bangalore
B. Math.(Hons.) I Year, Second Semester
Semestral Examination
Probability Theory-II

Time: 3 hours | May 5, 2010 | Instructor: B.Rajeev

Maximum Marks 60

1. The servicing of a machine requires two separate steps, with the time needed for the first step being an exponential random variable with mean .2 hours and the time for the second step being an independent exponential random variable with mean .3 hours. If a repair person has 20 machines to service,find an approximate value of the probability that all the work can be completed in 8 hours. [6]

2. A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the mean lifetime of this type of component is 100 hours and its standard deviation is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation for the next 2000 hours is at least.95? [6]

3. Let X be a normal random variable with parameters $\mu = 0$ and $\sigma^2 = 1$ and let I be a random variable, independent of X, such that $P\{I = 1\} = \frac{1}{2} = P\{I = 0\}$. Now define Y by

$$Y = \begin{cases} X & if \;\; I = 1 \\ -X & if \;\; I = 0 \end{cases}$$

a) Are X and Y independent? [3]

b) Show that Y is normal with mean 0 and variance 1. [4]

c) Show that $\mathrm{Cov}(X, Y) = 0$. [3]

4. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed continuous random variables. Let $N \geq 2$ be such that

$$X_1 \geq X_2 \geq \cdots \geq X_{N-1} < X_N$$

That is, N is the point at which the sequence stops decreasing. Show that $E[N] = e$. [8]

5. Let $U_1, U_2, \cdots$ be a sequence of independent uniform $(0, 1)$ random variables.

$$N(x) = min \left\{ n : \sum_{i=1}^{n} U_i > x \right\}$$

a) Show by induction on n that for $0 < x \leq 1$ and all $n \geq 0$,

$$P\{N(x) \geq n+1\} = \frac{x^n}{n!}$$

[5]

b) Show that $(E\ N(x)) = e^x$ [5]

6. If Z is a standard normal random variable and if Y is defined by $Y = a + bZ + cZ^2$, show that

$$\rho(Y, Z) = \frac{b}{\sqrt{b^2 + 2c^2}}$$

. [7]

7. Let $X_1, X_2, \cdots X_n$ be independent and identically distributed random variables and let $R = X_{(n)} - X_{(1)}$ be the range of the sample. If $X_1 \sim U(0,1)$ show that R has density

$$f(r) = \begin{matrix} n(n-1)r^{n-2}(1-r) & 0 \leq r \leq 1 \\ 0 & otherwise \end{matrix} .$$

[8]

8. Let $X_1, X_2, \cdots$ be independent and identically distributed random variables and N be a non negative integer valued random variable independent of the sequence $X_1, X_2, \cdots$. Let $Y = \sum_{i=1}^{N} X_i$.

a) Show that $M_Y(t) = E(M_{X_1}(t))^N$.

b) Show that $Var(Y) = EN\ Var(X_1) + (EX_1)^2\ Var\ (N)$. [4+6]